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ON SOME QUESTIONS CONCERNING PERMANENTS OF (1, − 1)-MATRICES

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ABSTRACT

Let Ω_n denote the set of all $n \times n$-$(1, -1)$-matrices. E.T.H. Wang has posed the following problem: For each $n \geqq 4$, can one always find nonsingular $A \in \Omega_n$ such that $|\operatorname{per} A| = |\det A|$ (*)? We present a solution for $n \leqq 6$ and, more generally, we show that (*) does not hold if $n = 2^k - 1$, $k \geqq 2$, even for singular $A \in \Omega_n$. Moreover, we prove that $\operatorname{per} A \neq 0$ if $A \in \Omega_n$, $n = 2^k - 1$, and we derive new results concerning the divisibility of the permanent in Ω_n by powers of 2.

1. Introduction. Preliminaries

Let Ω_n denote the set of all $n \times n$-$(1, -1)$-matrices and let $\tilde{\Omega}_n$ be the subset of all regular matrices in Ω_n.

We call $A, B \in \Omega_n$ equivalent, $A \sim B$, if B can be obtained from A by a sequence of the following operations:

(i) interchanging any two rows or any two columns;

(ii) transposition;

(iii) negating any row or any column.

Obviously, $\sim$ is an equivalence relation and $A \sim B$ implies

$$|\det A| = |\det B|, \tag{1}$$

$$|\operatorname{per} A| = |\operatorname{per} B|. \tag{2}$$

In general, the converse is not true (for a counterexample see [7], p. 354).

In [7], p. 360, Problem 1, E. T. H. Wang asks whether there is an $A \in \tilde{\Omega}_n$ satisfying

$$|\operatorname{per} A| = |\det A| \tag{3}$$

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for each $n \geqq 4$. (Cf. also the section "Conjectures and Unsolved Problems — A Current List" in H. Minc's book [3], p. 158, Problem 7.) In Section 2 we give an answer to this question for $n \leqq 6$ and for each $n = 2^k - 1$, $k \geqq 2$. In Section 3 we present an essential improvement of Wang's result on the divisibility of per A in Ω_n by powers of 2.

The following notations are used throughout our paper. If $A = (a_{kl})$ is an $n \times n$-matrix then $A(i \mid j)$ denotes the $(n-1)\times(n-1)$-submatrix obtained from A by deleting the i-th row and the j-th column. If σ is a permutation of $\{1,\cdots,n\}$ then $a_{1\sigma(1)}\cdot\cdots\cdot a_{n\sigma(n)}$ is called a diagonal product of A. $\pi(A)$ and $\nu(A)$ denote the numbers of positive and negative diagonal products in A, respectively. $\mu(A)$ is the number of negative entries in A and $\rho(n)$ is defined by

$$\rho(n) = \left[\frac{(n-1)^2}{2}\right].$$

2. The equation $|\operatorname{per} A| = |\det A|$ in Ω_n

Lemma 1. *Each $A \in \Omega_n$ is equivalent to one of the following matrices:*

$$B_1 = \left[\begin{array}{c|ccc} 1 & 1 & \cdots & 1 \\ \hline 1 & & & \\ \vdots & & C_1 & \\ 1 & & & \end{array}\right] \tag{4}$$

with $C_1 \in \Omega_{n-1}$, $\mu(C_1) \leqq \rho(n)$, or

$$B_2 = \left[\begin{array}{c|ccc} -1 & 1 & \cdots & 1 \\ \hline 1 & & & \\ \vdots & & C_2 & \\ 1 & & & \end{array}\right] \tag{5}$$

with $C_2 \in \Omega_{n-1}$, $\mu(C_2) \leqq \rho(n)$.

Proof. Suppose $A \sim B$ where B has the structure given by (4) but $\mu(B(1 \mid 1)) > \rho(n)$. Then we negate the second, third, $\cdots$, and n-th rows of B and after that the first column. These operations yield a matrix $B' \sim A$ with the structure given by (5) and with $\mu(B'(1 \mid 1)) \leqq \rho(n)$. Analogously we conclude in the converse case. □

Remark 1. For $B_i \in \tilde{\Omega}_n$, $i = 1,2$, we have

$$\mu(B_i(1 \mid 1)) \geqq n-1. \tag{6}$$

REMARK 2. For odd n, $B_1 \sim B_2$ holds in Lemma 1 if and only if $\mu(B_1(1 \mid 1)) = \rho(n)$.

Because of the invariance of (3) with respect to $\sim$ (cf. (1) and (2)) it suffices to consider a complete set of representatives modulo $\sim$ in $\tilde{\Omega}_n$. Lemma 1 and (6) enable us to exclude numerous undesired and equivalent matrices in our proofs.

PROPOSITION 1. *For* $2 \leqq n \leqq 4$ *no* $A \in \tilde{\Omega}_n$ *satisfies* (3).

PROOF. A detailed discussion of the occurring cases leads to the following sets of representatives. The assertion follows immediately from Table 1.

Table 1

n	$A \in \tilde{\Omega}_n$	$\lvert \operatorname{per} A \rvert$	$\lvert \det A \rvert$
2	$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$	0	2
3	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$	2	4
	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$	0	8
4	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$	4	8
	$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$	8	16

□

PROPOSITION 2. *Modulo* $\sim$ *there is exactly one* $A \in \tilde{\Omega}_5$ *which satisfies* (3), *namely*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

PROOF. We proceed as in the proof of Proposition 1 and get the assertion from Table 2.

$$A_1=\begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1 & 1\\ 1 & -1 & -1 & 1 & 1\\ 1 & -1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1 & -1\end{bmatrix},\quad A_2=\begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1 & 1\\ 1 & -1 & -1 & 1 & 1\\ 1 & -1 & 1 & -1 & 1\\ 1 & 1 & 1 & 1 & -1\end{bmatrix},$$

$$A_3=\begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 1 & -1 & -1 & 1 & 1\\ 1 & -1 & 1 & 1 & 1\\ 1 & 1 & 1 & -1 & 1\\ 1 & 1 & 1 & 1 & -1\end{bmatrix},\quad A_4=\begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 1 & -1 & -1 & -1 & 1\\ 1 & -1 & -1 & 1 & -1\\ 1 & -1 & 1 & 1 & 1\\ 1 & 1 & -1 & 1 & 1\end{bmatrix},$$

$$A_5=\begin{bmatrix} -1 & 1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1 & 1\\ 1 & 1 & -1 & 1 & 1\\ 1 & 1 & 1 & -1 & 1\\ 1 & 1 & 1 & 1 & -1\end{bmatrix},\quad A_6=\begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 1 & -1 & -1 & 1 & 1\\ 1 & -1 & 1 & 1 & 1\\ 1 & 1 & 1 & -1 & -1\\ 1 & 1 & 1 & -1 & 1\end{bmatrix},$$

$$A_7=\begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1 & 1\\ 1 & 1 & -1 & 1 & 1\\ 1 & 1 & 1 & -1 & 1\\ 1 & 1 & 1 & 1 & -1\end{bmatrix}.$$

Table 2

m	$\lvert \operatorname{per} A_m \rvert$	$\lvert \det A_m \rvert$
1	0	16
2	0	32
3	8	16
4	8	16
5	8	48
6	16	16
7	24	16

□

REMARK 3. The matrices A_3 and A_4 occurring in the proof of Proposition 2 are not equivalent. This follows from an equivalence test provided by H. Perfect ([4], p. 234, lemma 4.5). Therefore, the given set of representatives modulo $\sim$ of $\bar{\Omega}_5$ is complete as well as non-redundant.

Obviously, the matrix A_6 is equivalent to

$$S_5 := \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

which has an essentially simpler structure than A_6 and is easier to generalize. It is reasonable to ask whether for $n \geqq 6$ some of the matrices $S_n = (s_{ij})$ defined by

$$s_{ij} = \begin{cases} -1 & \text{for } 1 \leqq i \leqq j \leqq n \\ +1 & \text{otherwise} \end{cases}$$

satisfy (3). For an answer we need the following result on per S_n.

LEMMA 2. *Let S_n be defined as above. Then the following relations hold for each $n \geqq 1$:*

$$\begin{cases} \operatorname{per} S_{2n-1} = -\dfrac{2^{2n-1}(2^{2n}-1)B_{2n}}{n} \\ \operatorname{per} S_{2n} \;\;= 0 \end{cases} \tag{7}$$

where B_n denotes the n-th Bernoulli number.

PROOF. We define a matrix $S_n(t) = (s_{ij}(t))$ by

$$s_{ij}(t) = \begin{cases} t & \text{for } 1 \leqq i \leqq j \leqq n, \\ 1 & \text{otherwise.} \end{cases}$$

Considering the definition of the hit polynomial $A_n(t)$ of the so-called "triangular board" $S_n(t)$, one recognizes that it coincides with per $S_n(t)$ (see, e.g., J. Riordan [6], p. 165). By [6], p. 215, $A_n(t)$ has the exponential generating function

$$\sum_{n=0}^{\infty} \frac{A_n(t)}{n!} x^n = \frac{1-t}{1-t\exp[x(1-t)]}. \tag{8}$$

If we choose $t = -1$ and insert $S_n(-1) = S_n$ then (8) and $A_n(t) = \operatorname{per} S_n(t)$ imply

$$\sum_{n=0}^{\infty} \frac{\operatorname{per} S_n}{n!} x^n = \frac{2}{1+\exp(2x)}. \tag{9}$$

A connection between per S_n and the n-th Euler polynomial $E_n(y)$ can be established in the following way. By Abramowitz–Stegun [1], p. 804, formula 23.1.1,

$$\frac{2\exp(ys)}{1+\exp s}=\sum_{n=0}^{\infty}\frac{E_n(y)}{n!}s^n.$$

Setting $y=0$ and $s=2x$, we get

$$\frac{2}{1+\exp(2x)}=\sum_{n=0}^{\infty}\frac{E_n(0)}{n!}\cdot 2^n x^n. \tag{10}$$

From (9) and (10) we conclude

$$\sum_{n=0}^{\infty}\frac{\operatorname{per} S_n}{n!}x^n=\sum_{n=0}^{\infty}\frac{2^n E_n(0)}{n!}x^n$$

and hence

$$\operatorname{per} S_n=2^n E_n(0) \tag{11}$$

for all n. $E_n(0)$ admits the representation

$$E_n(0)=-\frac{2(2^{n+1}-1)B_{n+1}}{n+1}$$

where B_n is the n-th Bernoulli number (see [1], p. 805, formula 23.1.20). Since $B_{2n+1}=0$, we get the assertion from (11). □

REMARK 4. The result $\operatorname{per} S_{2n}=0$ is already contained in Wang's paper ([7], p. 359, example 2).

PROPOSITION 3. *For each $n \geqq 6$, S_n does not satisfy* (3).

PROOF. It can be easily shown that $\det S_n=2^{n-1}$. By Lemma 2 the assertion holds for each S_{2n}. Since

$$|B_{2n}|>\frac{2(2n)!}{(2\pi)^{2n}},$$

(7) implies

$$|\operatorname{per} S_{2n-1}|>\left(\frac{2}{\pi}\right)^{2n}(2n-1)!.$$

We are left to show the validity of the inequality

$$\left(\frac{2}{\pi}\right)^{2n}(2n-1)!>2^{(2n-1)-1}$$

for all $n \geqq 4$, or, if we define

$$\Phi(n) := \frac{4(2n-1)!}{\pi^{2n}},$$

$$\Phi(n) > 1 \tag{12}$$

for all $n \geqq 4$. This relation holds for $n = 4$ because $\Phi(4) > 2{,}124$. Assume that (12) is true for all $n \leqq k$, $k \geqq 4$. Then we have

$$\Phi(k+1) = \frac{4(2k+1)!}{\pi^{2k+2}} = \frac{4(2k-1)!}{\pi^{2k}} \cdot \frac{2k(2k+1)}{\pi^2} > \Phi(k) \cdot 1 > 1. \qquad \square$$

REMARK 5. The construction of a complete set of representatives modulo $\sim$ of $\tilde{\Omega}_6$ is rather troublesome, even if we take notice of the possible simplifications. We mention only that there are several non-equivalent matrices $A \in \tilde{\Omega}_6$ with $|\operatorname{per} A| = |\det A| = 32$ such as

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix}.$$

The relatively "frequent" appearance of $A \in \tilde{\Omega}_6$ with this property encouraged us to search for appropriate $A \in \tilde{\Omega}_7$ which satisfy (3), but we failed. An answer to this problem will be provided by Theorem 1. Before stating it we need some preliminary results. First we mention an important lemma due to Perfect ([4], p. 230, Corollary 3.4).

LEMMA 3. *Let $A \in \Omega_n$. Then*

$$\pi(A) \equiv 0(2^{n-[\log_2 n]-1}).$$

LEMMA 4. *For $n = 2^k - 1$ there is no $A \in \Omega_n$ with* $\operatorname{per} A = 0$.

PROOF. It is well known (see e.g. P. Bachmann [2], p. 52, formula (36)) that $2^{\kappa(n)}$ with $\kappa(n) = \sum_{i=1}^{\infty} [n/2^i]$ is the highest power of 2 which divides $n!$. For $n = 2^k - 1$,

$$\kappa(n) = n - [\log_2 n] - 1$$

and hence

$$n! = c(n) \cdot 2^{n-[\log_2 n]-1}$$

where $c(n)$ is an odd number depending only on n.

Suppose that an $A \in \Omega_n$ with $\operatorname{per} A = 0$ exists. Then we have

$$\pi(A) = \nu(A) = \tfrac{1}{2}n! = c(n) \cdot 2^{n-[\log_2 n]-2}$$

which contradicts Lemma 3.

REMARK 6. In [7], p. 358, remark 3, it is stated that for $n \equiv 3(4)$ it is nc known, in general, whether there exist $A \in \Omega_n$ such that $\operatorname{per} A = 0$. Lemma gives a partial answer to this problem.

LEMMA 5. *Let* $A \in \Omega_n$, $n = 2^k - 1$. *Then*

$$\operatorname{per} A \equiv 0(2^{n-[\log_2 n]-1}),$$

but

$$\operatorname{per} A \not\equiv 0(2^{n-[\log_2 n]}).$$

PROOF. As in the proof of Lemma 4 we get

$$n! = c_1(n) \cdot 2^{n-[\log_2 n]-1}$$

and by Lemma 3 we have

$$\pi(A) = c_2(A) \cdot 2^{n-[\log_2 n]-1}$$

for each $A \in \Omega_n$ where $c_2(A)$ is an integer depending only on A. Hence

$$\operatorname{per} A = 2\pi(A) - n! = (2c_2(A) - c_1(n)) \cdot 2^{n-[\log_2 n]-1}$$

where $2c_2(A) - c_1(n)$ is odd. (By Lemma 4, $\operatorname{per} A$ cannot vanish.) The assertic follows.

THEOREM 1. *For* $n = 2^k - 1$, $k \geqq 2$, *no* $A \in \Omega_n$ *satisfies* (3).

PROOF. For $k = 2$, i.e. $n = 3$, see Proposition 1. Let $k \geqq 3$, i.e. $n \geqq 7$. Lemma 5,

$$\operatorname{per} A = c(A) \cdot 2^{n-[\log_2 n]-1}$$

with an odd $c(A)$. (By Lemma 4, $\operatorname{per} A$ cannot vanish.) We have

$$n-[\log_2 n]-1<n-1$$

for all $n \geqq 7$. Because $\det A$ is always divisible by 2^{n-1} (see e.g. S. Reich [5], p. 650), (3) can never be satisfied. □

REMARK 7. Since $\tilde{\Omega}_n \subset \Omega_n$, Theorem 1 gives a negative answer to Wang's problem for an infinite number of n's. However, we do not know anything about the cases $n \neq 2^k-1$, $n \geqq 8$.

3. The divisibility of per A in Ω_n by powers of 2

PROPOSITION 4. *Let* $A \in \Omega_n$. *Then*

$$\operatorname{per} A \equiv 0(2^{n-[\log_2 n]-1}).$$

PROOF. The assertion follows from the relation $\operatorname{per} A = 2\pi(A)-n!$ and Lemma 3. □

REMARK 8. Since

$$n-[\log_2 n]-1 \geqq [n/2]$$

for each $n \geqq 5$ (for $n \geqq 9$ the inequality holds strictly), Proposition 4 is an essential improvement of proposition 1 in [7], p. 354.

A slightly more accurate statement is given by the following proposition.

PROPOSITION 5. *Let* $n \neq 2^k-1$. *Then for each* $A \in \Omega_n$,

$$\operatorname{per} A \equiv 0(2^{n-[\log_2 n]}). \tag{13}$$

For $n=2^k-1$ *see Lemma* 5.

PROOF. $n \neq 2^k-1$ implies

$$n! = c_1(n)\cdot 2^{n-[\log_2 n]}.$$

Further, by Lemma 3 we have

$$\pi(A) = c_2(A)\cdot 2^{n-[\log_2 n]-1}$$

and hence

$$\operatorname{per} A = 2\pi(A)-n! = (c_2(A)-c_1(n))\cdot 2^{n-[\log_2 n]}.$$

REMARK 9. The constant $c_1(n)$ appearing in the proof of Proposition 5 need not be odd, e.g.

$$c_1(2^k) \equiv 0(2^{k-1}).$$

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